A General Linear Sequential Filter

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A Kalman filter that accounts for time-correlated noise without augmenting the state vector or differencing the measurements is derived. The resulting filter equations demand less computer storage space and computation time than previously derived equations based upon state augmentation and measurement differencing. Solving the matrix Wiener-Hopf equation under the constraint that the noise states are not solved for gives the required formulation. For completeness the derivation also emphasizes the treatment in the estimation process of unsolved-for model parameters in the dynamic model and in the measurement model. Rather than solving for certain model parameters, the statistics of the modeling errors are accounted for in the filter weighting. Although this approach is not strictly optimal, computer sizing requirements are less when compared to an optimal filter that necessitates solving for the model parameters. The equations degenerate to the standard Kalman filter form when it is assumed that the modeling errors are nonexistent and that the measurement noise is white.

Nomenclature

A $\equiv \partial h/\partial T|X = \hat{X}_{n/n-1}, T = \hat{T}$
B $\equiv E(\delta\hat{T}\delta\hat{T}^T)$
C_n $\equiv E(\delta\hat{X}_{n/n}N_n^T)$
D $\equiv \partial\phi/\partial p|X = \hat{X}_{n-1/n-1}, p = \hat{p}$
E = the expectation operator
H $\equiv \partial h/\partial X|X = \hat{X}_{n/n-1}, T = \hat{T}$
h = functional relationship between the noise-free observations and the state
K = linear filter gain matrix
L_n $\equiv E(\delta\hat{X}_{n/n}\delta\hat{T}^T)$
N_n = measurement noise vector
$P_{i/k}$ $\equiv E(\delta\hat{X}_{i/k}\delta\hat{X}_{i/k}^T)$
p = a vector of the state dynamic parameters
$\hat{p}$ = an estimate of p
$\delta\hat{p}$ $\equiv p - \hat{p}$
Q $\equiv E(\mu_n\mu_n^T)$
S $\equiv E(\delta\hat{p}\delta\hat{p}^T)$
T = a vector of the measurement dynamics parameters
$\hat{T}$ = an estimate of T
$\delta\hat{T}$ $\equiv T - \hat{T}$
t_n = value of time at n times the time increment plus the initial value of time
V_n $\equiv E(\delta\hat{X}_{n/n}\delta\hat{p}^T)$
W_n = a gaussian white noise vector
$\delta\hat{X}_{i/k}$ = an estimate of X_i based on measurements up to and including those at t_k, $k \leq i$
$\hat{X}_{i/k}$ = $X_i - \hat{X}_{i/k}$
X_n = the state vector at time t_n
Z_n = the measurement vector at t_n
$\hat{Z}_n$ = an estimate of Z_n based on measurements up to and including those at t_{n-1}
$\delta\hat{Z}_n$ $\equiv Z_n - \hat{Z}_n$
μ = a gaussian white noise vector
ρ = the correlation coefficient relating N_{n-1} to N_n
Σ_n = $E\{[N_n - E(N_n)][N_n - E(N_n)]^T\}$
Φ = $\partial\phi/\partial X|X = \hat{X}_{n-1/n-1}, p = \hat{p}$
ϕ = transition function relating the state at time t_n to its past value at time t_{n-1}

Introduction

IN the past, two methods have in general been employed to account for correlated measurement noise in a linear sequential (Kalman) filter. The first method, proposed by Kalman[1] and known as the state augmentation method, involves the construction of a hypothetical shaping filter whose input is white noise and whose output is noise of the required correlations. Here the dynamics of the shaping filter are included as a part of the system dynamics, and its states are estimated along with the system states. The second method, advanced by Bryson and Henrikson,[2,3] requires that the most recent past value of the measurement vector, weighted by the noise correlation coefficient matrix, be subtracted from the present measurement vector. In this way the correlated components of the measurement noise vector subtract out, giving an effective measurement with only white noise errors.

There are, however, some disadvantages inherent in both of these methods. In the first method, state augmentation requires an increase in the state vector dimensions and a corresponding increase in the dimensions of the state estimate error covariance matrix. In practice, such dimensional increase may be prohibitive because of restricted computer storage space. There have been instances in the past, for example, where filters based on white noise assumptions were resorted to because of computer limitations. The second method alleviates some of the difficulties inherent in the state augmentation scheme, since the state vector does not require expansion to accommodate the measurement noise components. In view of this, the measurement differencing method thus effects an economy in computer storage requirements. However, the differencing method itself presents some disadvantages that in some cases can be bothersome in a practical implementation. For instance, it requires storage of the measurement vector for a complete computation cycle and also storage of the matrix of partial derivatives relating the measurements to the state. More importantly, the primary inconvenience arises when several measurement sets are available at one time, and it is desired to process each set separately (see Appendix A). Ordinarily the standard white noise form of the Kalman filter would present no problem; however, if the measurements of each set all contain time-correlated noise, it is necessary to invert portions of the transition matrix in order to propagate the state vector estimate back to the prior time point. This adds to the complexity of the matrix arithmetic in the computer.

A sequential filter is derived in this paper that accounts for correlated measurement noise in a way that requires neither the adjoining of a shaping filter to the filter dynamic model nor the differencing of successive measurements prior to applying the filter weights. The filter equations are obtained by solving the matrix Wiener-Hopf equations which results in a filter that is optimum in a minimum variance sense; however, the filter is constrained to be a linear estimator, and it is

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assumed that each measurement set is processed without differencing with previous measurement sets. The first constraint implies no restrictions if the random errors are gaussian and of small enough amplitude so that all perturbations are within the bounds of system linearity. A filter derived with the second constraint degenerates to a white noise filter as the correlation coefficient of the measurement errors goes to zero; and when the correlation coefficient goes to unity, it degenerates to a filter which does not estimate a measurement bias error but accounts for its effect.

Also of considerable importance is the portion of the derivation dealing with the effects of unsolved-for modeling errors. These errors reflect the limits in human ability to describe the real world in exact mathematical terms. It is assumed for the purposes of this paper that the system can be adequately represented by a linearization with respect to the model parameters being considered as significant error sources. This differs significantly from the situation where the mathematical form itself is uncertain. Inclusion of these parameters in the solution vector may be prohibited by computer space availability, but ignoring their effects can cause a divergent estimate. Such is the case, for example, in an orbit determination of a 24-hr satellite when uncertainties in certain geopotential harmonic coefficients are ignored. Serious filter estimate divergence has also been observed in radio-guided launch missions when uncertainties in engine parameters such as the effective exhaust velocity are not properly accounted for. Other examples of modeling errors are radar location survey uncertainties and gyro drift rates.

Previous attempts to reduce the divergent effects of the dynamic modeling errors have depended upon the addition of an artificial "white noise" covariance matrix to the state error covariance matrix, but such a scheme must be performed empirically on a trial and error basis. On the other hand, the modeling errors are accounted for in the general filter equations of this paper in a systematic manner, having been obtained from the solution of the Wiener-Hopf equation. These results have been derived previously and are presented by Schlee et al.[4] in a slightly different form. By explicitly computing the cross-correlations between the state estimate errors and the modeling errors during each computation cycle, an accounting for the effect of the modeling errors is maintained.

Derivation

Let the dynamics of the state be written as the nonlinear vector equation

$$X_n = \phi(X_{n-1},p) + \Gamma\mu_n \tag{1}$$

Let the measurements be related to the state by the nonlinear vector equation

$$Z_n = h(X_n,T) + N_n \tag{2}$$

where N_n is assumed to be generated by a Markov process of the form

$$N_n = \rho N_{n-1} + qW_n \tag{3}$$

It is assumed for the purposes of the development of the filter equations that all inputs to the system described by Eq. (1) are independent of the filter estimates; that is, the filter can only observe the system by means of the measurements Z_n.

Assume that the system can be modeled by

$$\hat{X}_{n/n-1} = \phi(\hat{X}_{n-1/n-1},\hat{p}) \tag{4}$$

and

$$\hat{Z}_n = h(\hat{X}_{n/n-1},\hat{T}) \tag{5}$$

The errors in the estimates of Eqs. (4) and (5) are from the definitions

$$\delta\hat{X}_{n/n-1} = \phi(X_{n-1},p) - \phi(\hat{X}_{n-1/n-1},\hat{p}) + \Gamma\mu_n \tag{6}$$

$$\delta\hat{Z}_n = h(X_n,T) - h(\hat{X}_{n/n-1},\hat{T}) + N_n \tag{7}$$

For estimates close to the true values, Eqs. (6) and (7) may be expanded in a Taylor series to get

$$\delta\hat{X}_{n/n-1} = \Phi\delta\hat{X}_{n-1/n-1} + D\delta\hat{p} + \Gamma\mu_n \tag{8}$$

$$\delta\hat{Z}_n = H\delta\hat{X}_{n/n-1} + A\delta\hat{T} + N_n \tag{9}$$

It will be assumed that the filter is of the form

$$\hat{X}_{n/n} = \hat{X}_{n/n-1} + K_n\delta\hat{Z}_n \tag{10}$$

It has been shown by Kalman[1] that Eq. (10) results in an optimal estimate of the state with respect to a quadratic loss function when Eqs. (8) and (9) are valid and when $\delta\hat{Z}_n$, $\delta\hat{X}_{n/n-1}$ and $\delta\hat{X}_{n-1/n-1}$ are gaussian. In many applications, therefore, Eq. (10) can be called an optimum linear filter.

Given all measurements up to and including the one at t_n, the estimation error at t_n is, through use of Eq. (9),

$$\delta\hat{X}_{n/n} = (I - K_nH)\delta\hat{X}_{n/n-1} - K_nA\delta\hat{T} - K_nN_n \tag{11}$$

The Wiener-Hopf equation (see Appendix B) specifies that the filter gain K_n is given by

$$K_n = E(\delta\hat{X}_{n/n-1}\delta\hat{Z}_n{}^T)[E(\delta\hat{Z}_n\delta\hat{Z}_n{}^T)]^{-1} \tag{12}$$

From Eqs. (6) and (7)

$$E(\delta\hat{X}_{n/n-1}\delta\hat{Z}_n{}^T) = P_{n/n-1}H^T + \Phi L_{n-1}A^T + \Phi C_{n-1}\rho^T \tag{13}$$

Also

$$E(\delta\hat{Z}_n\delta\hat{Z}_n{}^T) = HP_{n/n-1}H^T + \Sigma_n + ABA^T + H\Phi L_{n-1}A^T + AL_{n-1}{}^T\Phi^TH^T + H\Phi C_{n-1}\rho^T + \rho C_{n-1}{}^T\Phi^TH^T \tag{14}$$

When Eqs. (14) and (13) are combined according to Eq. (12), the filter gain becomes

$$K_n = (P_{n/n-1}H^T + \Phi L_{n-1}A^T + \Phi C_{n-1}\rho^T)(HP_{n/n-1}H^T + ABA^T + H\phi L_{n-1}A^T + AL_{n-1}{}^T\phi^TH^T + \Sigma_n + H\phi C_{n-1}\rho^T + \rho C_{n-1}{}^TH^T)^{-1} \tag{15}$$

provided that the inverse exists. Note that when B and the correlation terms L and C are zero, Eq. (15) degenerates to the standard white noise Kalman filter gain.

It is next necessary to obtain expressions for the terms P, C, V, and L in Eq. (15). Through use of Eq. (8) and after the required expectations are performed, the propagated state estimate covariance matrix becomes

$$P_{n/n-1} = \Phi P_{n-1/n-1}\Phi^T + \Gamma Q\Gamma^T + DSD^T + \Phi V_{n-1}D^T + DV_{n-1}{}^T\Phi^T \tag{16}$$

The last three terms of Eq. (16) account for the uncertainty in the dynamic model parameter vector p. The last two terms which account for the correlation of the state estimate error and the parameter uncertainty would be omitted in a so-called "state noise" compensation scheme. These two terms tend to prevent overcompensation by DSD^T by subtracting from DSD^T that covariance in $P_{n/n-1}$ already accounted for in prior computations of the state covariance matrix by the filter. As a result the filter gains will be smaller than they would be without the correlation terms of Eq. (16) and less measurement noise will get through to the estimates.

From Eq. (11) the state estimate covariance matrix after the measurement set at time t_n is processed is

$$P_{n/n} = (I - K_nH)P_{n/n-1}(I - K_nH)^T + K_nABA^TK_n{}^T - (I - K_nH)\Phi L_{n-1}A^TK_n{}^T - K_nAL_{n-1}{}^T\Phi^T(I - K_nH)^T + K_n\Sigma_nK_n{}^T - (I - K_nH)\Phi C_{n-1}\rho^TK_n{}^T - K_n\rho C_{n-1}{}^T\Phi^T(I - K_nH)^T \tag{17}$$

Also from Eqs. (11) and (3)

$$C_n = E\{[(I - K_nH)\Phi\delta\hat{X}_{n-1/n-1} + (I - K_nH)D\delta\hat{p} + (I - K_nH)\Gamma\mu_n - K_nN_n - K_nA\delta\hat{T}](\rho N_{n-1} + qW_n)^T\} = (I - K_nH)\Phi C_{n-1}\rho^T - K_n\Sigma_n \quad (18)$$

The V_n is obtained as

$$V_n = E\{[(I - K_nH)\Phi\delta\hat{X}_{n-1/n-1} + (I - K_nH)\Gamma\mu_n + (I - K_nH)D\delta\hat{p} - K_nA\delta\hat{T} - K_nN_n](\delta\hat{p})^T\} = (I - K_nH)\Phi V_{n-1} + (I - K_nH)DS \quad (19)$$

Finally

$$L_n = (I - K_nH)\Phi L_{n-1} - K_nAB \quad (20)$$

To summarize, the recursive equations of the filter are

$$P_{n/n-1} = \Phi P_{n-1/n-1}\Phi^T + \Gamma Q\Gamma^T + DSD^T + \Phi V_{n-1}D^T + DV_{n-1}{}^T\Phi^T \quad (21)$$

$$\Sigma_n = \rho\Sigma_{n-1}\rho^T + q\Sigma_w q^T \quad (22)$$

$$K_n = (P_{n/n-1}H^T + \Phi L_{n-1}A^T + \Phi C_{n-1}\rho^T)(HP_{n/n-1}H^T + ABA^T + H\Phi L_{n-1}A^T + AL_{n-1}{}^T\Phi^TH^T + \Sigma_n + H\Phi C_{n-1}\rho^T + \rho C_{n-1}{}^T\Phi^TH^T)^{-1} \quad (23)$$

$$\hat{X}_{n/n-1} = \phi(\hat{X}_{n-1/n-1},\hat{p}) \quad (24)$$

$$\hat{X}_{n/n} = \hat{X}_{n/n-1} + K_n[Z_n - h(\hat{X}_{n/n-1},\hat{T})] \quad (25)$$

$$P_{n/n} = (I - K_nH)P_{n/n-1}(I - K_nH)^T + K_nABA^TK_n{}^T - (I - K_nH)\Phi L_{n-1}A^TK_n{}^T - K_nAL_{n-1}{}^T\Phi^T(I - K_nH)^T + K_n\Sigma_nK_n{}^T - (I - K_nH)\Phi C_{n-1}\rho^TK_n{}^T - K_n\rho C_{n-1}{}^T\Phi^T(I - K_nH)^T \quad (26)$$

$$C_n = (I - K_nH)\Phi C_{n-1}\rho^T - K_n\Sigma_n \quad (27)$$

$$V_n = (I - K_nH)\Phi V_{n-1} + (I - K_nH)DS \quad (28)$$

$$L_n = (I - K_nH)\Phi L_{n-1} - K_nAB \quad (29)$$

Since the model parameters considered in Eqs. (21–29) are not explicitly solved for, their uncertainties remain constant throughout the filtering process. These uncertainties, then, when propagated into the filter estimates, cause larger estimate errors than would have been the case had the parameters been part of the state vector. Had these parameters been included in the state vector, their uncertainties would decrease with time and consequently affect the other estimates less. The only advantage in using the above equations, then, would be reduced complexity and computer size requirements. It is impossible to specify precisely the computer storage and execution times for each opposing method since these quantities depend upon the particular problem involved and its particular implementation. However, a qualitative comparison of the two methods can be obtained by specifying, for instance, the difference in the number of multiplication operations performed.

Suppose that in one filter the state vector contained 13 elements, one of which is a model parameter. A second filter omits this model parameter from the state vector and thereby has only 12 state vector elements. Equations (21–29) are utilized to implement the second filter and account for the uncertainty in the omitted model parameter. If the measurement vector dimension is three in both cases, the second filter requires 270 less multiplications.

Appendix A

With several measurement sets are available at one time point and it is desirable to process each set separately, an inconvenient state transition matrix inversion is required. This is demonstrated in the following development.

The differencing filters described by Bryson and Henrikson[2,3] difference the measurement vector in such a way that the correlations of the measurement noise cancel, and the noise of the effective measurement is white. At the nth time point the effective measurement is

$$Y_n = Z_n - \rho Z_{n-1} \quad (A1)$$

Let ξ_n and ζ_n be subsets of Z_n such that

$$Z_n = \begin{bmatrix}\xi_n\\ \zeta_n\end{bmatrix} = \begin{bmatrix}h_1(X_n)\\ h_2(X_n)\end{bmatrix} \quad (A2)$$

If the errors of ξ_n and ζ_n are independent, Eq. (A1) then becomes

$$Y_n = \begin{bmatrix}\eta_n\\ \omega_n\end{bmatrix} = \begin{bmatrix}\xi_n\\ \zeta_n\end{bmatrix} - \begin{bmatrix}\rho_1 & 0\\ 0 & \rho_2\end{bmatrix}\begin{bmatrix}\xi_{n-1}\\ \zeta_{n-1}\end{bmatrix} \quad (A3)$$

Suppose that it is desirable to obtain an estimate of the state by first processing η_n and then to refine that estimate by processing ω_n. The filter equations then become

$$\bar{X}_{n/n} = \phi(\hat{X}_{n-1/n-1}) + K_1(\eta_n - \hat{\eta}_n) \quad (A4)$$

and

$$\hat{X}_{n/n} = \bar{X}_{n/n} + K_2(\omega_n - \hat{\omega}_n) \quad (A5)$$

where K_1 and K_2 = colored noise Kalman filter gains, $(\hat{\ })$ = an estimate of the quantity ( ), and $\bar{X}_{n/n}$ = an estimate of $\hat{X}_n$, given all measurements up to the nth cycle and given η_n. The measurement estimate $\hat{\eta}_n$ is given by

$$\hat{\eta}_n = h_1[\phi(\hat{X}_{n-1/n-1})] - \rho_1h_1(\hat{X}_{n-1/n-1}) \quad (A6)$$

The measurement estimate $\hat{\omega}_n$ is computed from $\bar{X}_{n/n}$ as

$$\hat{\omega}_n = h_2(\bar{X}_{n/n}) - \rho_2h_2[\phi^{-1}(\bar{X}_{n/n})] \quad (A7)$$

The inversion of the transition function is required to obtain an estimate of ξ_{n-1} from $\bar{X}_{n/n}$.

Similarly the matrix H_a relating the incremental differenced measurement to the state can be expressed as

$$H_a = \begin{pmatrix}H_{a1}\\ H_{a2}\end{pmatrix}$$

where

$$H_a = \partial Y/\partial X|_{X_n=\hat{X}_{n/n-1}}$$

$$H_{a1} = \partial\eta/\partial X|_{X_n=\hat{X}_{n/n-1}}$$

$$H_{a2} = \partial\omega/\partial X|_{X_n=\hat{X}_{n/n-1}}$$

Then

$$H_{a1} = H_{1n}\Phi - \rho_1H_{1(n-1)} \quad (A8)$$

$$H_{a2} = H_{2n} - \rho_2H_{2(n-1)}\Phi^{-1} \quad (A9)$$

It is seen in Eq. (A9) that an inversion of the transition matrix is required.

Appendix B

One of the means of obtaining the Wiener-Hopf equation, which is useful in deriving Kalman filter equations, is given here.

Define the inner product of two vectors a and b to be

$$\langle a,b\rangle = X_1{}^TE(ab^T)X_2 \quad (B1)$$

where E is the expectation vector, a is an n vector, and b is an m vector. The X_1 is any n vector and X_2 any m vector such that

$$X_1 \subset X_2 \quad \text{for} \quad n < m$$

$$X_1 \supset X_2 \quad \text{for} \quad m < n$$

$$X_1 = X_2 \quad \text{for} \quad m = n$$

The requirements of an inner product for real vectors are as follows:

$$\langle a + c,b\rangle = \langle a,b\rangle + \langle c,b\rangle$$

$$\langle \alpha a,b\rangle = \alpha\langle a,b\rangle$$

$$\langle a,a\rangle \geq 0; \quad \langle a,a\rangle = 0 \Longrightarrow a = 0$$

That Eq. (B1) satisfies the first two requirements can be trivially shown. The third requirement is merely the condition that a covariance matrix be positive definite.

Let ξ be the Hilbert space containing the state variable X_n and ζ be the subspace spanned by the measurements $Z_1,\ldots,Z_n$. The difference

$$\delta\hat{X}_{n/n} = X_n - \hat{X}_{n/n}$$

then is contained in that subspace of ξ, not in ζ; i.e., $\xi - \zeta$ is orthogonal to ζ. Therefore, every vector in the subspace $\xi - \zeta$ is orthogonal to every vector in ζ. The vector difference

$$\delta\hat{Z}_n = Z_n - \hat{Z}_n$$

is contained in ζ. Therefore, $\delta\hat{Z}_n$ is the orthogonal to $\delta\hat{X}_{n/n}$ and

$$\langle\delta\hat{X}_{n/n}, \quad \delta\hat{Z}_n\rangle = X_1{}^T E(\delta\hat{X}_{n/n}\delta\hat{Z}_n{}^T)X_2 = 0 \tag{B2}$$

$$E(\delta\hat{X}_{n/n}\delta\hat{Z}_n{}^T) = 0 \tag{B3}$$

Substituting Eq. (10) of the text into Eq. (B3) results in

$$E[(\delta\hat{X}_{n/n-1} - K_n\delta\hat{Z}_n{}^T)\delta\hat{Z}_n{}^T] = 0 \tag{B4}$$

$$K_n = E(\delta\hat{X}_{n/n-1}\delta\hat{Z}_n{}^T)[E(\delta\hat{Z}_n\delta\hat{Z}_n{}^T)]^{-1} \tag{B5}$$

which is the Wiener-Hopf equation.

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A New Treatment of Roll-Pitch Coupling for Ballistic Re-Entry Vehicles

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A new source of roll torques which can explain anomalous re-entry motions is presented. Geometric nose asymmetries are shown to cause a normal force component not dependent on angle of attack. The influence of the blunt nose on hypersonic aerodynamics is described with a simple, analytic theory—including effects of entropy gradient and dynamic pressure deficit. This Embedded-Newtonian theory is used to predict load distributions from nose-ablation asymmetries, yielding trim normal forces for high bluntness and trim moments for small bluntness. A powerful coupling with mass asymmetries is thus provided without requiring large trim angles of attack. The resulting motion is predicted with equilibrium trim solutions and verified with digital simulations.

Nomenclature

A = reference area $(\pi d^2/4)$, ft^2
a = acceleration, g's
b = $\left[C_{N\alpha}\left(1 - \frac{I_x}{I}\right) - \frac{md^2}{I}(C_{m_q} + C_{m\dot{\alpha}})\right]\frac{\bar{q}A}{mV\omega}$
C_A = axial force coefficient
C_D = drag force coefficient
C_l = roll moment coefficient
C_{l_o} = surface-induced roll moment coefficient
C_{l_g} = roll damping due to grooves/rad
C_{l_f} = roll damping due to friction/rad
C_N = normal force coefficient
$\Delta^i C_{N_o}$ = nose asymmetry induced normal force independent of α
C_m = pitching moment coefficient
C_{mo} = trim moment coefficient
C_p = pressure coefficient
C_{po} = blast wave pressure coefficient
$C_{p\text{Newt}}$ = Newtonian pressure coefficient
d = base diameter, ft
I = pitch moment of inertia, slug-ft^2
I_x = roll moment of inertia, slug-ft^2
i = $(-1)^{1/2}$
L_N = total surface recession at stagnation point, ft
L_o = sharp cone length, ft
m = vehicle mass, slugs
p = spin rate, rad/sec
$\bar{q}$ = dynamic pressure $(\rho V^2/2)$, lb/ft^2
$\bar{q}_l$ = local dynamic pressure, lb/ft^2
r = radial distance from center line, ft
R = complex angle-of-attack moduli
t = time, sec

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